# M2201 Sheet 1. Solutions.

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1.
$$\begin{aligned}
1254 &= 1110 \cdot 1 + 144 \\
1110 &= 144 \cdot 7 + 102 \\
144 &= 102 \cdot 1 + 42 \\
42 &= 18 \cdot 2 + 6 \\
18 &= 6 \cdot 3 + 0 \\
\gcd(1254, 1110) &= 6
\end{aligned}$$

Running the algorithm backwards,

$$\begin{aligned}
6 &= 42 - 18 \cdot 2 \\
&= 42 - (102 - 42 \cdot 2) \cdot 2 = 5 \cdot 42 - 2 \cdot 102 \\
&= 5 \cdot (144 - 102) - 2 \cdot 102 = 5 \cdot 144 - 7 \cdot 102 \\
&= 5 \cdot 144 - 7 \cdot (1110 - 7 \cdot 144) = 54 \cdot 144 - 7 \cdot 1110 = \\
&= 54 \cdot (1254 - 1110) - 7 \cdot 1110 \\
&= 54 \cdot 1254 - 61 \cdot 1110
\end{aligned}$$

Therefore $h = 54, k = -61$.

2. The equation $1254x + 1110y = 18$ has solutions as 6 divides 18. A particular solution is $x_0 = 162$, $y_0 = -183$ and the general solution is $x = 162 + 185n, y = -183 - 209n$ with $n$ any integer.

   The equation $1254x + 1110y = 15$ does not have solutions as 6 does not divide 15.

3. Suppose $a$ is not prime. Write $a = a_1a_2$ with $a_1, a_2 \neq 1$. Clearly $a$ divides $a_1a_2$. As $a_1 < a$, $a$ does not divide $a_1$. By the property satisfied by $a$, $a$ divides $a_2$ but this is not possibe because $a_2 < a$. Hence $a$ is prime.

4. (i) One has $n = (n+1)(n!+1) - ((n+1)!+1)$. As $d$ divides $n!+1$ and $(n+1)!+1$, $d$ divides $n$.

   (ii) $1 = (n!+1) - n(n-1)!$. As $d$ divides $n!+1$ and $n$, $d = 1$.

5. $d$ divides $2(2a+b)+(-1)(a+2b)=3a$ and $d$ divides $(-1)(2a+b)+2(a+2b)=3b$. As $a$ and $b$ are coprime, $3=3ah+3bk$ hence $d$ divides 3. Therefore $d=1$ or 3.

# M2201 Sheet 2. Solutions.

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1. For any $2 \leq i \leq n$, $i$ clearly divides $n! + i$ and $1 < i < n! + i$, hence the $n! + i$ are composite.

2. First observe that any integer is of the form $6k$, $6k+1$, $6k+2$, $6k+3$, $6k+4$ or $6k+5$ but only integers of the form $6k+1$ or $6k+5$ can be prime (in all other cases you can divide by 2 or 3).

   Suppose there are finitely many primes of the form $6k+5$, list them $p_1, \cdots, p_r$ with $p_1 = 5$. Consider
   $$Q = 6p_2 \cdots p_r + 5$$
   $Q$ is not divisible by 5 because otherwise 5 would divide $6p_2 \cdots p_r$ which is does not. Similarly $Q'$ is not divisible by $p_i$, $i > 1$.

   We'll therefore get a contradiction if we show that $Q$ is divisible by some prime of the form $6k+5$.

   We do it by induction. The smallest integer $6k+5$ is 5 which is prime and there is nothing to prove. Suppose that this holds for all integers of the form $6k+5$ which are $< Q$. If $Q$ is prime then there is nothing to prove. Suppose that $Q = Q_1 Q_2$ with $1 < Q_i < Q$. We need to show that one of the $Q_i$s is $\equiv 5 \mod 6$. They can only be $\equiv 5$ or $\equiv 1 \mod 6$ ($Q$ is not divisible by 2 or 3).

   Suppose $Q_i \equiv 1 \mod 6$. Then $Q = Q_1 Q_2 \equiv 1 \mod 6$ contradicting that $Q \equiv 5 \mod 6$. It follows that some $Q_i$ and hence $Q$ has a prime factor $\equiv 5 \mod 6$ i.e. some $p_i$. This yields a contradiction.

   The method does not work for $6k+1$ since when you multiply two integers of the form $6k+5$, you get an integer of the form $6k+1$...

3. - [i] One finds $\{[3], [8], [14]\}$
   - [ii] One finds $\{[21]\}$

4. $154 = 22 \times 7$. If $a$ and 23 are coprime, 23 being prime, by Fermat's little theorem, we have
   $$a^{22} \equiv 1 \mod 23$$
   Hence $a^{154} \equiv 1^7 \equiv 1 \mod 23$. Therefore 23 divides $a^{154} - 1$.

5.

$$\begin{aligned}
37 &= 2 \times 17 + 3 \\
17 &= 5 \times 3 + 2 \\
3 &= 1 \times 2 + 1 \\
1 &= 3 - 2 \\
&= 3 - (17 - 5 \times 3) \\
&= 6 \times 3 - 17 \\
&= 6(37 - 2 \times 17) - 17 \\
&= 6 \times 37 - 13 \times 17.
\end{aligned}$$

Therefore

$$z \equiv 6 \times 37 \times 4 - 13 \times 17 \times 3 \equiv 225 \bmod 629.$$

# M2201 Sheet 3. Solutions.

Andrei Yafaev

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1. (i)

$$\begin{aligned} f(x) &= (x^2+x+1)g(x)+(x+1) \\ g(x) &= (2x+1)(x+1)+2. \end{aligned}$$

Therefore $\mathrm{hcf}(f,g)=1$.

(ii)

$$\begin{aligned} 2 &= g(x)-(2x+1)(x+1) \\ &= g(x)-(2x+1)(f(x)-(x^2+x+1)g(x)) \\ &= (1+(2x+1)(x^2+x+1))g(x)-(2x+1)f(x) \\ &= (2x^3+3x^2+3x+2)g(x)-(2x+1)f(x). \end{aligned}$$

Therefore

$$1=\left(x^3+\frac{3}{2}x^2+\frac{3}{2}x+1\right)g(x)+\left(-x-\frac{1}{2}\right)f(x),$$

so we can take $h(x)=-x-\frac{1}{2}$ and $k(x)=x^3+\frac{3}{2}x^2+\frac{3}{2}x+1$.

2.

$$\begin{aligned} a(x) &= (x+2)b(x)+(2x^2+3x+4) \\ b(x) &= (3x+4)(2x^2+3x+4)+2x \\ 2x^2+3x+4 &= (x+4)(2x)+4. \end{aligned}$$

Therefore $d=1$.

$$\begin{aligned} 4 &= (2x^2+3x+4)-(x+4)(2x) \\ &= (2x^2+3x+4)-(x+4)(b(x)-(3x+4)(2x^2+3x+4)) \\ &= (1+(x+4)(3x+4))(2x^2+3x+4)-(x+4)b(x) \\ &= (3x^2+x+2)(2x^2+3x+4)-(x+4)b(x) \\ &= (3x^2+x+2)(a(x)-(x+2)b(x))-(x+4)b(x) \\ &= (3x^2+x+2)a(x)-((x+4)+(3x^2+x+2)(x+2))b(x) \\ &= (3x^2+x+2)a(x)-(3x^3+2x^2+3)b(x) \\ &= (3x^2+x+2)a(x)+(2x^3+3x^2+2)b(x). \end{aligned}$$

Dividing by 4:

$$1 = (2x^2 + 4x + 3)a(x) + (3x^3 + 2x^2 + 3)b(x).$$

Therefore $h(x) = 2x^2 + 4x + 3$ and $k(x) = 3x^3 + 2x^2 + 3$.

3. We have $x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1)$. In $\mathbb{Q}[x]$ both polynomials are irreducible: $x^2 - 2$ is of degree 2 hence if it was reducible it would be a product of two polynomials of degree one i.e. it would have a root. But it does not as 2 is not square in $\mathbb{Q}$. Similarly for $x^2 + 1$. Hence $(x^2 - 2)(x^2 + 1)$ is the factorisation in $\mathbb{Q}[x]$.

   In $\mathbb{R}[x]$, $(x^2 - 2) = (x - \sqrt{2})(x + \sqrt{2})$ while $x^2 + 1$ is irreducible (it's of degree 2 and $-1$ is not a square in $\mathbb{R}$). The factorisation is $(x - \sqrt{2})(x + \sqrt{2})(x^2 + 1)$.

   In $\mathbb{C}[x]$, the factorisation is $(x - \sqrt{2})(x + \sqrt{2})(x - i)(x + i)$.

   In $\mathbb{F}_2[x]$,
   $$x^4 - x^2 - 2 = x^4 - x^2 = x^2(x^2 - 1) = x^2(x - 1)^2$$

   In $\mathbb{F}_3[x]$. $-2 = 1$ hence $x^2 - 2 = x^2 + 1$ hence $x^4 - x^2 - 2 = (x^2 + 1)^2$. It suffices to factorise $x^2 + 1$ in $\mathbb{F}_3[x]$. One immediately checks that $0, 1$ and $2$ are not roots hence this polynomial is irreducible. The factorisation is $(x^2 + 1)^2$.

   In $\mathbb{F}_5[x]$, once checks that $0, 1, 2, 3, 4$ are not roots of $x^2 - 2$ hence it is irreducible. 2 is a root of $x^2 + 1$ and 3 is another root, hence $x^2 + 1 = (x - 2)(x - 3)$ in $\mathbb{F}_5$. The factorsiation is $(x^2 - 2)(x - 2)(x - 3)$.

4. (i) By Fermat's little theorem, any element of $\mathbb{F}_p$ is a root of $x^{p-1} - 1$.
   Hence $x(x - 1) \cdots (x - (p - 1))$ divides $x^p - x$ :

   $$x^{p-1} - 1 = x(x - 1) \cdots (x - (p - 1))q$$

   As the degrees are equal, $q \in \mathbb{F}_p$ and by comparing terms of highest degrees, we find that $q = 1$.

   (ii) Exaluate the equality in (i) at $x = 0$. We get

   $$-1 = (-1)^{p-1}(p - 1)!$$

   in $\mathbb{F}_p$. Now, in $\mathbb{F}_p$, $(-1)^{p-1} = 1$. The result follows.

# M2201 Sheet 4. Solutions.

Andrei Yafaev

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1. (i) Let $v \in V$. Write $u = T(v)$ and $w = v - T(v)$. We have $v = u + w$ and $T(w) = T(v) - T^2(v) = T(v) - T(v) = 0$ hence $w \in \ker(T)$. It follows that $V = \ker(T) + \mathrm{im}(T)$.

   Let $y \in \ker(T) \cap \mathrm{im}(T)$, say $y = T(x)$. Then $T(y) = 0 = T^2(x) = T(x) = y$, hence $\ker(T) \cap \mathrm{im}(T) = \{0\}$.

   This shows that $V = \ker(T) \oplus \mathrm{im}(T)$.

   The converse does not hold. Take $T = 2Id$. Then $\ker(T) = \{0\}$ and $\mathrm{im}(T) = V$ hence $V = \ker(T) \oplus \mathrm{im}(T)$ but $T^2 \neq T$.

   (ii) Let $y \in \mathrm{im}(T)$, then $y = T(x)$ for some $x \in V$. We have $T(y) = T^2(x) = 0$ hence $y \in \ker(T)$.

   The converse also holds. Suppose $\mathrm{im}(T) \subset \ker(T)$. Let $v \in V$, then $T(v) \in \mathrm{im}(T) \subset \ker(T)$ hence $T^2(v) = T(T(v)) = 0$, hence $T^2 = 0$.

2. One finds

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   It is obvious that the dimension of the image is 3 with basis - column vectors. Hence $\mathrm{im}(T) = span\{x, x^2, x^3\}$. The dimensionn of the kernel is then zero and $\ker(T) = \{0\}$.

3. The map $tr\colon M_n(k) \longrightarrow k$ is a linear map. It is surjective (for example $\lambda E_{1,1}$ has trace $\lambda$) hence the rank is one. We need to find $\dim\ker(tr)$. Rank-nullity theorem shows that $\dim\ker(tr) = n^2 - 1$.

4. For any matrix $M$ we can write (when $k = \mathbb{R}$),

$$M = \frac{1}{2}(M + M^t) + \frac{1}{2}(M - M^t)$$

   Now, $\frac{1}{2}(M + M^t)$ is in $S$ and $\frac{1}{2}(M - M^t)$ is in $A$. It follows that $V = S + A$.

Let $M \in S \cap A$. Then $M_{i,j} = M_{j,i} = -M_{i,j}$ hence $2M_{i,j} = 0$ and as $k = \mathbb{R}$, we have $S \cap A = \{0\}$. The sum is direct.

This does not hold if $k = \mathbb{F}_2$ because then any symmetric matrix is antisymmetric, hence $S = A$, the sum $S + A$ is not direct.

5. Let

   - $\{v_1, \ldots, v_p\}$ be a basis for $U \cap W$.
   - Complete to a basis of $U$ : $\{v_1, \ldots, v_p, u_1, \ldots, u_q\}$
   - .. and to a basis of $W$ : $\{v_1, \ldots, v_p, w_1, \ldots, w_r\}$

   Clearly $\{v_1, \ldots, v_p, u_1, \ldots, u_q, w_1, \ldots, w_r\}$ is a family of vectors of $U + W$.

   We need to show that they are linearly independent.

$$\sum_i \alpha_i v_i + \sum \beta_j u_j + \sum_k \gamma_k w_k = 0$$

   Then

$$\sum \beta_j u_j \in U \cap W$$

   hence

$$\sum \beta_j u_j - \sum \beta'_l v_l = 0$$

   because $\{v_1, \ldots, v_p, u_1, \ldots, u_q\}$ is a basis of $U$, we get that all $\beta_j = 0$. Similarly all $\gamma_k = 0$ and last all $\alpha_i = 0$.

   Next we need to show that $\{v_1, \ldots, v_p, u_1, \ldots, u_q, w_1, \ldots, w_r\}$ generates $U+W$. This is easy : let $u + w$ in $U + W$, expand $u$ and $w$ in the above bases for $U$ and $W$...

   This finishes the proof.

# M2201 Sheet 5. Solutions.

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1. Since these matrices are both upper triangular we can write down their characteristic polynomials:

$$\mathrm{ch}_A(X) = (X-1)^2(X-2), \quad \mathrm{ch}_B(X) = (x-1)(x-2)^2.$$

Then $(A-I)(A-2I) = 0 = (B-I)(B-2I)$ so $m_A(x) = m_B(x) = (X-1)(X-2)$. $A$ has eigenvalues 1 and 2. For the eigenvalue 1 we have two linearly independent eigenvectors $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $v_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}$. For the eigenvalue 2 we have an eigenvector $v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$ So we obtain

$$P = \begin{pmatrix}1 & 0 & 1\\0 & 0 & 1\\0 & 1 & 0\end{pmatrix}, \quad P^{-1}AP = \begin{pmatrix}1 & & \\ & 1 & \\ & & 2\end{pmatrix}.$$

For $B$ we also have eigenvalues 1 and 2. For the eigenvalue 1 we have an eigenvector $v_1 = \begin{pmatrix}0\\-2\\1\end{pmatrix}$. For the eigenvalue 2 we have two linearly independent eigenvectors $v_2 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and $v_3 = \begin{pmatrix}0\\1\\0\end{pmatrix}$.

So we obtain

$$Q = \begin{pmatrix}0 & 1 & 0\\-2 & 0 & 1\\1 & 0 & 0\end{pmatrix}, \quad Q^{-1}BQ = \begin{pmatrix}1 & & \\ & 2 & \\ & & 2\end{pmatrix}.$$

2. Differentiating a polynomial reduces the degree, so if we differentiate an element of $V$ eleven times we have 0. Therefore $D^{11} = 0$.

$m_D$ is a factor of $X^{11}$. Hence $m_D(X) = X^b$ for some $b \leq 11$. It follows that 0 is the only zero of $m_D$ hence 0 is the only eigenvalue.

$$\begin{aligned}
V_1(0) &= \left\{f \in V : \frac{\mathrm{d}f}{\mathrm{d}x} = 0\right\} = \mathrm{Span}\{1\}, \\
V_2(0) &= \left\{f \in V : \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = 0\right\} = \mathrm{Span}\{1, x\}, \\
V_3(0) &= \left\{f \in V : \frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = 0\right\} = \mathrm{Span}\{1, x, x^2\}.
\end{aligned}$$

3. Calculation shows that $ch_A(x) = (x-2)^2(x-3)$.

   The matrix is diagonalisable if and only if $m_A(x) = (x-2)(x-3)$.

   We calculate:
   $$(A - 2I_3)(A - 3I_3) = \begin{pmatrix} 0 & 0 & 0 \\ -a & -a & a \\ -a & -a & a \end{pmatrix},$$

   It follows that $A$ is diagonalisable if and only if $a = 0$.

4. As $\dim(Im(T)) = 1$, $\dim(ker(T)) = n - 1$ (rank-nullity theorem).

   Let $v \neq 0$ be an element of $Im(T)$. Notice that because $\dim(Im(T)) = 1$, $v$ is a basis of $Im(T)$.

   Suppose $T(v) = 0$. Let $u \in V$, then $T(u) = \lambda v$ for some $\lambda \in k$. Then
   $$T^2(u) = \lambda T(v) = 0$$
   hence $T^2 = 0$.

   If $T(v) \neq 0$, then $T(v) = \lambda v$ for some $\lambda \neq 0$ and $v$ is an eigenvector attached to $\lambda$.

   Choose a basis $\{e_1, \ldots, e_{n-1}\}$ of $ker(T)$. Of course all $e_i$s are eigenvectors attached to eigenvalue 0. We claim that $\{e_1, \ldots, e_{n-1}, v\}$ is a basis of $V$.

   It suffices to show that they are linearly independent: write
   $$\alpha v + \sum \alpha_i e_i = 0$$

   Apply $T$, one finds $\alpha \lambda v = 0$ which implies that $\alpha = 0$.

   Then $\sum \alpha_i e_i = 0$ implies that $\alpha_i = 0$ for all $i$ because the $e_i$s are linearly independent.

   As $\{e_1, \ldots, e_{n-1}, v\}$ is a basis of $V$ and it consists of eigenvectors, $T$ is diagonalisable.

# M2201 Sheet 5. Solutions.

Andrei Yafaev

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1. i False. Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

   ii False. Take $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $m_A = ch_A = x^2$ but it's not diagonalisable.

   iii True. Eigenvalues are roots of the minimal polynomial $m_T$. Let $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues. Then $(x-\lambda_1)\cdots(x-\lambda_n)$ divides $m_T$. But $\deg(m_T) \leq n = \deg(ch_T)$ hence $m_T = (x-\lambda_1)\cdots(x-\lambda_n)$ and $T$ is diagonalisable.

   iv False. Take $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$. 0 is the only eigenvalue but $m_T = x(x^2+1)$.

   v True. The only root of $m_T$ is $\lambda$, by fundamental theorem of algebra, $m_T(x) = (x-\lambda)^b$ for some integer $b$.

2. i As $T$ is invertible, 0 is not eigenvalue hence $m_T(0) \neq 0$. This means that $x$ does not divide $m_T$. As $x$ is irreducible, $x$ and $m_T$ are coprime.

   ii Bézout's identity: there exist polynomials $h, k$ such that $xh + km_T(x) = 1$. Using $m_T(T) = 0$, we get $Th(T) = Id$ hence $T^{-1} = h(T)$.

3. i We have $T^2 = Id$, hence the minimal polynomial divides $x^2 - 1$. As $T \neq \pm Id$, the minimal polynomial is $x^2 - 1$.
   $x^2 - 1 = (x+1)(x-1)$ diagonalisable over $\mathbf{R}$ and $\mathbf{C}$, not over $\mathbf{F}_2$.

   ii Over $\mathbf{R}$, $T$ is diagonalisable, the eigenvalues are $\pm 1$. The eigenspace corresponding to 1 is the space of symmetric matrices, the eigenspace corresponding to $-1$ is that of antisymetric matrices. We see that

$$ch_T(x) = (x-1)^{n(n+1)/2}(x+1)^{n(n-1)/2}$$

4. The matrix of $T$ is

$$\begin{pmatrix} 2 & 0 & 0 & 6 & 0 \\ 0 & 2 & 0 & 0 & 24 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

We see that $ch_T = (x-2)^5$. One finds that $(T-2I)^2 = 0$ and $T - 2I \neq 0$ hence $m_T(x) = (x-2)^2$.

Generalised eigenspaces:

$$V_1(2) = Span(1, x, x^2)$$

and $V_2(2) = V = \mathbf{R}[x]_4$.

Pre-Jordan basis: $B_1 = \{1, x, x^2\}$, $B_2 = \{x^3, x^4\}$.

Jordan basis: $(T-2I)x^3 = 6$ and $(T-2I)x^4 = 24x$. Substitute those for 1 and $x$. the jordan basis is:

$$\{6, x^3, 24x, x^4, x^2\}$$

Jordan normal form:

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

# M2201 Sheet 7. Solutions.

Andrei Yafaev

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1. i False.

   If such map existed then, because $m = (x-\lambda)^2$, the JNF would have two blocks: one $2 \times 2$ and one $1 \times 1$ hence $\dim V_1(\lambda)$ would be two.

   ii True. Take $V = \mathbf{C}^3$ and $T$ to be represented by the following matrix in the standard basis:

$$\begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

   iii True. $m_{T_1} = m_{T_2}$ is either $x - \lambda$ in which case $T_1 = T_2 = \lambda I$; or it is $(x-\lambda)^2$ in which case the Jordan normal form for both has one $2 \times 2$ block.

   iv True.

   Possibilities for $m_{T_1} = m_{T_2}$ are $x - \lambda$ (both $\lambda I$), $(x-\lambda)^2$ (JNF for both has one $2 \times 2$ block and one $1 \times 1$ block) or $(x-\lambda)^3$ (JNF for both has one $3 \times 3$ block).

   v False.

   JNF for $T_1$ can have two $2 \times 2$ blocks and JNF for $T_2$ one $2 \times 2$ and two $1 \times 1$. The minimal polynomials are the same but JNFs different.

2. $ch_A = (x-1)^3(x-2)$ hence 1 and 2 are eigenvalues.

$$V_1(2) = Span \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

   and

$$V_1(1) = Span\begin{pmatrix}2\\3\\0\\1\end{pmatrix}$$

There will be one $3\times 3$ block corresponding to eigenvalue 1. One already sees that $m_T = (x-1)^3(x-2)$.

One calculates the Jordan basis of $V_3(1)$:

$$\{\begin{pmatrix}2\\3\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}\}$$

3. i Because $\dim(V_1(2)) = 3$, JNF has 3 blocks corresponding to eigenvalue 2, they are all $1\times 1$ blocks.

   For eigenvalue $-1$, we have we have $\dim V_1(-1) = 2$ hence two blocks. We can have either two $2\times 2$ blocks, in which case $m = (x-2)(x+1)^2$; or one $3\times 3$ and one $1\times 1$ block, in which case $m = (x-2)(x+1)^3$.

   ii $\dim(V_1(0)) = 1$ hence one $2\times 2$ block corresponding to eigenvalue 0.

   $\dim(V_1(1)) = 4$ hence four blocks : three one by one blocks and one one two by two.

   We have $m_T = x^2(x-1)^2$.

4. If $i \geq b$, then clearly $V_i(\lambda) = V = V_{i+1}$.

   Suppose that $V_i(\lambda) = V_{i+1}(\lambda)$ for some $i$. Suppose that $i < b$ i.e. $i \leq b-1$.

   Let $v \in V_b(\lambda) = V$ such that $v \notin V_{b-1}(\lambda)$. Such a vector exists because $m_T = (x-\lambda)^b$ hence $V_{b-1}(\lambda) \neq V$.

   Then $b-(i+1) \geq 0$. Consider $w = (T-\lambda I)^{b-(i+1)}v$. We have $(T-\lambda)^{i+1}v = (T-\lambda)^b v = 0$ hence $w \in V_{i+1}(\lambda) = V_i(\lambda)$.

   Hence $(T-\lambda I)^i w = 0$. But $(T-\lambda I)^i w = (T-\lambda I)^{b-1}v$. Hence $v \in V_{b-1}(\lambda)$, a contradiction. Hence $i \geq b$.

# M2201 Sheet 8. Solutions.

Andrei Yafaev

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1. i False.
   Look at $f\colon k \times k \longrightarrow k$ such that $f(x, y) = xy^2$.

   ii True.

   iii True.

   iv False.
   Let $f$ be the bilinear form on $\mathbf{R}^2$ given by $f(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}) = x_1x_2 - y_1y_2$

   Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $f(v, v) = 0$ and $v \neq 0$ is in $Span(v) \cap v^{\perp}$.

2. Let $v \in V$. If $v = 0$ then the conclusion is obviously true.

   Suppose $v \neq 0$. Consider $T\colon V \longrightarrow k$ defined by $T(w) = f(v, w)$. It follows from the assumption on $f$ that $T$ is a non-zero linear map.

   We have $\{v\}^{\perp} = \ker(T)$ and $rank(T) = 1 = \dim(Span(v))$. The equality follows from rank-nullity theorem.

3. Notice that $f(p, q) = (pq)''(0)$. Clearly bilinear, symmetric.

   One finds the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   By double row-column operations, one finds the canonical form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

   The rank is three, signature $(2, 1)$.

4. In the standard basis, the matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Canonical form:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Rank is 4, signature $(3, 1)$.

5. One finds orthogonal basis (the three column vectors):

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & -1 & -3 \\ 0 & 0 & 1 \end{pmatrix}$$

In this basis the matrix is:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

and canonical form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Rank is 3, signature $(1, 2)$.